Part I Problems

Problem 1: Find the general solution by separation of variables:

$$\frac{dy}{dx} = 2 - y, \qquad y(0) = 0$$

Problem 2: Find the general solution by separation of variables:

$$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2}$$

Problem 3: The rate of change of a certain population is proportional to the square root of its size. Model this situation with a differential equation.

Problem 4:

The rate of change of the velocity of an object is proportional to the square of the velocity. Model this situation with a differential equation.

Problem 5:

In a population of fixed size S, the rate of change of the number N of persons who have heard a rumor is proportional to the number of those who have not yet heard it. Model this situation with a differential equation.

Problem 6: The amount of a certain medicine in the bloodstream decays exponentially with a half-life of 5 hours. In order to keep a patient safe during a one-hour procedure, there needs to be at least 50 mg of medicine per kg of body weight. How much medicine should be administered to a 60kg patient at the start of the procedure?

Problem 7: Early one morning it starts to snow. At 7AM a snowplow sets off to clear the road. By 8AM, it has gone 2 miles. It takes an additional 2 hours for the plow to go another 2 miles. Let $t = 0$ when it begins to snow, let x denote the distance traveled by the plow at time t. Assuming the snowplow clears snow at a constant rate in cubic meters/hour:

a) Find the DE modeling the value of x.

b) When did it start snowing?

Problem 8: A tank holds 100 liters of water which contains 25 grams of salt initially. Pure water then flows into the tank, and salt water flows out of the tank, both at 5 liters/minute. The mixture is kept uniform at all times by stirring.

a) Write down the DE with IC for this situation.

b) How long will it take until only 1 gram of salt remains in the tank?

Part II Problems

Problem 1: [Natural growth, separable equations] In recitation a population model was studied in which the natural growth rate of the population of oryx was a constant $k > 0$, so that for small time intervals Δt the population change $x(t + \Delta t) - x(t)$ is well approximated by $kx(t)\Delta t$. (You also studied the effect of hunting them, but in this problem we will leave that aside.) Measure time in years and the population in kilo-oryx (ko).

A mysterious virus infects the oryxes of the Tana River area in Kenya, which causes the growth rate to decrease as time goes on according to the formula $k(t) = k_0/(a + t)^2$ for $t \geq 0$, where a and k_0 are certain positive constants.

(a) What are the units of the constant a in "$a + t$," and of the constant k_0?

(b) Write down the differential equation modeling this situation.

(c) Write down the general solution to your differential equation. Don't restrict yourself to the values of t and of x that are relevant to the oryx problem; take care of all values of these variables. Points to be careful about: use absolute values in $\int \frac{dx}{x} = \ln|x| + c$ correctly, and don't forget about any "lost" solutions.

(d) Now suppose that at $t = 0$ there is a positive population x_0 of oryx. Does the progressive decline in growth rate cause the population stabilize for large time, or does it grow without bound? If it does stabilize, what is the limiting population as $t \to \infty$?

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1. $\frac{dy}{dx} = 2-y, \quad y(0)=0$

$$\frac{1}{2-y}\,dy = 1\,dx$$

$$\int \frac{1}{2-y}\,dy = \int 1\,dx$$

$$-\ln(2-y) + c = x$$

$$2-y = e^{-x+c}$$

$$y = -e^{-x+c} + 2$$

$$0 = -e^{c} + 2$$

$$e^{c} = 2$$

$$c = \ln 2$$

$$\therefore\ y = -e^{-x}\cdot e^{\ln 2} + 2$$

$$= 2(1-e^{-x})$$

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2. $$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2}$$

$$\int \frac{1}{(y-1)^2}\, dy = \int \frac{1}{(x+1)^2}\, dx$$

$$\int (y-1)^{-2}\, dy = \int (x+1)^{-2}\, dx$$

$$\frac{1}{-1}(y-1) \cdot \frac{y^2}{2}$$

$$-\frac{1}{2}(y-1)y^2$$

3. $\frac{dx}{dt} = k\sqrt{x}$, $\quad x = x(t)$

$k > 0$

4. $v = v(t)$

$\frac{dv}{dt} = kv^2$

5. $P = S$

$N = N(t)$

$\frac{dN}{dt} = k(S - N)$, $N < S$

6. Body weight $= w$

$\frac{e^{-k(0)}}{2} = e^{-k(5)}$

$$e^{-5k} = \frac{1}{2}$$

$$-5k = \ln 1 - \ln 2$$

$$k = \frac{\ln 2}{5}$$

$$\frac{dx}{dt} = -kx$$

$$\frac{dx}{dt} = -\frac{\ln 2}{5}x$$

$$\int \frac{1}{x}\,dx = \int -\frac{\ln 2}{5}\,dt$$

$$\ln x = -\frac{\ln 2}{5}t + c$$

$$x = Ce^{-\frac{\ln 2}{5}t}$$

At $t=1$, $x = 50 \times 60 = 3000$

$$3000 = Ce^{-\frac{\ln 2}{5}(1)}$$

$$C = 3000\,e^{\frac{\ln 2}{5}}$$

$$= 3446.10$$

$$= 3447$$

7. Let k_1 be the rate of snowfall and k_2 be the rate of clearing snow.

a)

$$\Rightarrow k_1 t \cdot \Delta x \approx k_2 \Delta t$$

$$\frac{\Delta x}{\Delta t} = \frac{k_2}{k_1 t}$$

$$\Rightarrow \frac{dx}{dt} = \frac{k}{t}$$

b) $$\int dx = \int \frac{k}{t}\, dt$$

$$x = k \ln|t| + C$$

$$\ln|t| = \frac{x - C}{k}$$

$$|t| = e^{\frac{1}{k}x - c}$$

$$t = Ce^{\frac{1}{k}x}$$

$$\Rightarrow T+1 = Ce^{\frac{2}{k}}, \quad T+3 = Ce^{\frac{4}{k}}$$

$$\Rightarrow \quad T+0 = C$$

$$T+1 = Ce^{\frac{2}{k}}$$

$$T+3 = Ce^{\frac{4}{k}}$$

$$\Rightarrow \quad C+1 = Ce^{\frac{2}{k}}$$

$$C+3 = Ce^{\frac{4}{k}}$$

$$C\left(e^{\frac{2}{k}} - 1\right) = 1$$

$$C = \frac{1}{e^{\frac{2}{k}} - 1}$$

8. $V = 100$, $x(0) = 25$

a) $\frac{dx}{dt} = 0 - 5\frac{x(t)}{V}$

$\Rightarrow \frac{dx}{dt} = -\frac{x(t)}{20}$

$\int \frac{1}{x(t)}\,dx = -\frac{1}{20}\int 1\,dt$

$\ln|x| = -\frac{t}{20}$

$x = Ce^{-\frac{t}{20}}$

$x(0) = 25$, $\therefore x = 25e^{-\frac{t}{20}}$

$25 = Ce^{0}$

$C = 25$

b) $1 = 25e^{-\frac{t}{20}}$

$-\frac{t}{20} = \ln\frac{1}{25}$

$\ln 25 = \frac{t}{20}$

$t = 20\ln 25$

$= 64.38$ minutes